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CONDITIONAL INTEGRAL TRANSFORMS AND CONVOLUTIONS OF BOUNDED FUNCTIONS ON AN ANALOGUE OF WIENER SPACE

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Abstract. Let $C[0,t]$ denote the function space of all real-valued continuous paths on $[0,t]$. Define $X_n : C[0,t] \to \mathbb{R}^{n+1}$ and $X_{n+1} : C[0,t] \to \mathbb{R}^{n+2}$ by $X_n(x) = (x(t_0), x(t_1), \cdots, x(t_n))$ and $X_{n+1}(x) = (x(t_0), x(t_1), \cdots, x(t_n),\ x(t_{n+1}))$, where $0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = t$. In the present paper, using simple formulas for the conditional expectations with the conditioning functions X_n and X_{n+1}, we evaluate the $L_p(1 \le p \le \infty)$-analytic conditional Fourier-Feynman transforms and the conditional convolution products of the functions which have the form

$$\int_{L_2[0,t]} \exp\{i(v,x)\}d\sigma(v) \int_{\mathbb{R}^r} \exp\bigg\{ i\sum_{j=1}^{r} z_j(v_j,x)\bigg\} d\rho(z_1,\cdots,z_r)$$

for $x \in C[0,t]$, where $\{v_1, \cdots, v_r\}$ is an orthonormal subset of $L_2[0,t]$ and σ and ρ are the complex Borel measures of bounded variations on $L_2[0,t]$ and $\mathbb{R}^r$, respectively. We then investigate the inverse transforms of the function with their relationships and finally prove that the analytic conditional Fourier-Feynman transforms of the conditional convolution products for the functions, can be expressed in terms of the products of the conditional Fourier-Feynman transforms of each function.

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1. Introduction and preliminaries

Let $C_0[0,t]$ denote the Wiener space, that is, the space of real-valued continuous functions x on the closed interval $[0,t]$ with $x(0)=0$. Chang and Skoug ([3]) introduced the concepts of conditional Fourier-Feynman transform and conditional convolution product on the Wiener space $C_0[0,t]$. In that paper, they examined the effects that drift has on the conditional Fourier-Feynman transform, the conditional convolution product, and various relationships that occur between them. Further works were studied by the author and his coauthors ([2, 9]). In fact, they ([2]) introduced the L_1-analytic conditional Fourier-Feynman transform and the conditional convolution product over Wiener paths in abstract Wiener space and then, established their relationships between them of certain cylinder type functions. The author ([9]) extended the relationships between the conditional convolution product and the $L_p(1 \le p \le 2)$-analytic conditional Fourier-Feynman transform of the same kind of cylinder functions. Moreover, on $C[0,t]$, the space of the real-valued continuous paths on $[0,t]$, Kim ([12]) extended the relationships between the conditional convolution product and the $L_p(1 \le p \le \infty)$-analytic conditional Fourier-Feynman transform of the functions in a Banach algebra which corresponds to the Cameron-Storvick's Banach algebra $\mathcal{S}$ ([1]). The author ([4, 5, 6]) established several relationships between the $L_p(1 \le p \le \infty)$-analytic conditional Fourier-Feynman transforms and the conditional convolution products of the cylinder functions on $C[0,t]$. In particular, he ([4, 5]) derived an evaluation formula for the L_p-analytic conditional Fourier-Feynman transforms and the conditional convolution products of the same cylinder functions with the conditioning functions $X_n : C[0,t] \to \mathbb{R}^{n+1}$ and $X_{n+1} : C[0,t] \to \mathbb{R}^{n+2}$ given by $X_n(x) = (x(t_0), x(t_1), \cdots, x(t_n))$ and $X_{n+1}(x) = (x(t_0), x(t_1), \cdots, x(t_n), x(t_{n+1}))$, where $0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = t$ is a partition of $[0,t]$, and then, derived their relationships. Note that X_n is independent of the present time t while X_{n+1} is wholly dependent on the present time. In this paper, we further develop the relationships in ([4, 5, 12]) on the more generalized space $(C[0,t], w_\varphi)$, an analogue of the Wiener space associated with the probability measure φ on the Borel class $\mathcal{B}(\mathbb{R})$ of $\mathbb{R}$ ([11, 13, 14]). For the conditioning functions X_n and X_{n+1}, we proceed to study the relationships between the conditional convolution products and the analytic conditional Fourier-Feynman transforms of bounded functions defined on $C[0,t]$. In fact, using simple formulas for the conditional w_φ-integrals given X_n and X_{n+1}, we evaluate the $L_p(1 \le p \le \infty)$-analytic conditional Fourier-Feynman transforms and the conditional convolution products for the functions of the form

$$(1.1)\quad \int_{L_2[0,t]} \exp\{i(v,x)\}d\sigma(v) \int_{\mathbb{R}^r} \exp\bigg\{ i\sum_{j=1}^{r} z_j(v_j,x)\bigg\} d\rho(z_1,\cdots,z_r)$$

for w_φ-a.e. $x \in C[0,t]$, where $\{v_1,\cdots,v_r\}$ is an orthonormal set in $L_2[0,t]$, σ is a complex Borel measure of bounded variation on $L_2[0,t]$ and ρ is a bounded complex Borel measure on $\mathbb{R}^r$. We then investigate various relationships between the conditional Fourier-Feynman transforms and the conditional convolution products of the functions given by (1.1). Finally, we show that the L_p-analytic conditional Fourier-Feynman transform of the conditional convolution product for the functions Ψ_1 and Ψ_2 given by (1.1), can be expressed by the formula

$$\begin{aligned} &T_q^{(p)}[[(\Psi_1 * \Psi_2)_q|X_n](\cdot,\vec{\xi}_n)|X_n](y,\vec{\zeta}_n) \\ &= \bigg[T_q^{(p)}[\Psi_1|X_n]\bigg(\frac{1}{\sqrt{2}}y, \frac{1}{\sqrt{2}}(\vec{\zeta}_n+\vec{\xi}_n)\bigg)\bigg] \\ &\quad\times \bigg[T_q^{(p)}[\Psi_2|X_n]\bigg(\frac{1}{\sqrt{2}}y, \frac{1}{\sqrt{2}}(\vec{\zeta}_n-\vec{\xi}_n)\bigg)\bigg] \end{aligned}$$

for a nonzero real q, w_φ-a.e. $y \in C[0,t]$ and P_{X_n}-a.e. $\vec{\xi}_n, \vec{\zeta}_n \in \mathbb{R}^{n+1}$. Thus the analytic conditional Fourier-Feynman transform of the conditional convolution product for the functions, can be interpreted as the product of the analytic conditional Fourier-Feynman transform of each function.

Let $\mathbb{C}$, $\mathbb{C}_+$ and $\mathbb{C}_+^\sim$ denote the set of complex numbers, the set of complex numbers with positive real parts and the set of nonzero complex numbers with nonnegative real parts, respectively.

Let $C = C[0,t]$ be the space of all real-valued continuous functions on the closed interval $[0,t]$ with its Borel class $\mathcal{B}(C[0,t])$. For a probability measure φ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, let w_φ be an analogue of the Wiener measure on $\mathcal{B}(C[0,t])$ associated with φ ([11, 13, 14]). Let $F : C[0,t] \to \mathbb{C}$ be integrable and X be a random vector on $C[0,t]$ assuming that the value space of X is a normed space equipped with the Borel σ-algebra. Then, we have the conditional expectation $E[F|X]$ of F given X from a well known probability theory. Furthermore, there exists a P_X-integrable $\mathbb{C}$-valued function ψ on the value space of X such that $E[F|X](x) = (\psi \circ X)(x)$ for w_φ-a.e. $x \in C[0,t]$, where P_X is the probability distribution of X. The function ψ is called the conditional w_φ-integral of F given X and it is also denoted by $E[F|X]$.

Throughout this paper, let $0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = t$ be a partition of $[0,t]$, where n is a positive integer. For any x in $C[0,t]$, define the polygonal function $[x]$ of x by

$$(1.2)\quad [x](s) = \sum_{j=1}^{n+1} \chi_{(t_{j-1},t_j]}(s)\bigg(\frac{t_j - s}{t_j - t_{j-1}} x(t_{j-1}) + \frac{s - t_{j-1}}{t_j - t_{j-1}} x(t_j)\bigg) + \chi_{\{t_0\}}(s)x(t_0)$$

for $s \in [0,t]$, where $\chi_{(t_{j-1},t_j]}$ and $\chi_{\{t_0\}}$ denote the indicator functions. Similarly, for $\vec{\xi}_{n+1} = (\xi_0, \xi_1, \cdots, \xi_{n+1}) \in \mathbb{R}^{n+2}$, define the polygonal function $[\vec{\xi}_{n+1}]$ of $\vec{\xi}_{n+1}$ by (1.2), where $x(t_j)$ is replaced by ξ_j for $j = 0, 1, \cdots, n+1$. Let $X_{n+1} : C[0,t] \to \mathbb{R}^{n+2}$ and $X_n : C[0,t] \to \mathbb{R}^{n+1}$ be given by

$$(1.3)\quad X_{n+1}(x) = (x(t_0), x(t_1), \cdots, x(t_n), x(t_{n+1}))$$

and

$$(1.4)\quad X_n(x) = (x(t_0), x(t_1), \cdots, x(t_n)),$$

respectively. For a function $F : C[0,t] \to \mathbb{C}$ and $\lambda > 0$, let $F^\lambda(x) = F(\lambda^{-\frac{1}{2}}x)$, $X^\lambda_{n+1}(x) = X_{n+1}(\lambda^{-\frac{1}{2}}x)$ and $X^\lambda_n(x) = X_n(\lambda^{-\frac{1}{2}}x)$, where X_{n+1} and X_n are given by (1.3) and (1.4), respectively. Suppose that $E[F^\lambda]$ exists for each $\lambda > 0$. By the definition of the conditional w_φ-integral and the equation (6) of Theorem 2.9 in [8],

$$E[F^\lambda|X^\lambda_{n+1}](\vec{\xi}_{n+1}) = E[F(\lambda^{-\frac{1}{2}}(x - [x]) + [\vec{\xi}_{n+1}])]$$

for $P_{X^\lambda_{n+1}}$-a.e. $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$, where $P_{X^\lambda_{n+1}}$ is the probability distribution of X^λ_{n+1} on $(\mathbb{R}^{n+2}, \mathcal{B}(\mathbb{R}^{n+2}))$. Throughout this paper, for $y \in C[0,t]$ let

$$I^\lambda_F(y, \vec{\xi}_{n+1}) = E[F(y + \lambda^{-\frac{1}{2}}(x - [x]) + [\vec{\xi}_{n+1}])]$$

unless otherwise specified, where the expectation is taken over the variable x. Moreover, we can obtain from (2.6) of Theorem 2.5 in [7]

$$(1.5)\quad E[F^\lambda|X^\lambda_n](\vec{\xi}_n) = \left[\frac{\lambda}{2\pi(t - t_n)}\right]^{\frac{1}{2}} \int_{\mathbb{R}} I^\lambda_F(0, \vec{\xi}_{n+1}) \exp\bigg\{-\frac{\lambda}{2}\frac{(\xi_{n+1} - \xi_n)^2}{t - t_n}\bigg\} d\xi_{n+1}$$

for $P_{X^\lambda_n}$-a.e. $\vec{\xi}_n = (\xi_0, \xi_1, \cdots, \xi_n) \in \mathbb{R}^{n+1}$, where $\vec{\xi}_{n+1} = (\xi_0, \xi_1, \cdots, \xi_n, \xi_{n+1})$ for $\xi_{n+1} \in \mathbb{R}$ and $P_{X^\lambda_n}$ is the probability distribution of X^λ_n on

$(\mathbb{R}^{n+1}, \mathcal{B}(\mathbb{R}^{n+1}))$. For $y \in C[0,t]$, let $K_F^\lambda(y, \vec{\xi}_n)$ be given by (1.5) where 0 is replaced by y. If $I_F^\lambda(0, \vec{\xi}_{n+1})$ has the analytic extension $J_\lambda^*(F)(\vec{\xi}_{n+1})$ on $\mathbb{C}_+$ as a function of λ, then it is called the conditional analytic Wiener w_φ-integral of F given X_{n+1} with parameter λ and denoted by $E^{anw_\lambda}[F|X_{n+1}](\vec{\xi}_{n+1}) = J_\lambda^*(F)(\vec{\xi}_{n+1})$ for $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$. Moreover, if for a nonzero real q, $E^{anw_\lambda}[F|X_{n+1}](\vec{\xi}_{n+1})$ has a limit as λ approaches to $-iq$ through $\mathbb{C}_+$, then it is called the conditional analytic Feynman w_φ-integral of F given X_{n+1} with parameter q and denoted by $E^{anf_q}[F|X_{n+1}](\vec{\xi}_{n+1}) = \lim_{\lambda\to -iq} E^{anw_\lambda}[F|X_{n+1}](\vec{\xi}_{n+1})$. Similarly, the definitions of $E^{anw_\lambda}[F|X_n](\vec{\xi}_n)$ and $E^{anf_q}[F|X_n](\vec{\xi}_n)$ are understood with $K_F^\lambda(0, \vec{\xi}_n)$ if X_{n+1} is replaced by X_n.

For a given extended real number p with $1 < p \le \infty$, suppose that p and p' are related by $\frac{1}{p} + \frac{1}{p'} = 1$(possibly $p' = 1$ if $p = \infty$). Let F_n and F be measurable functions such that $\lim_{n\to\infty} \int_C |F_n(x) - F(x)|^{p'} dw_\varphi(x) = 0$. Then we write $\text{l.i.m.}_{n\to\infty}(w^{p'})(F_n) = F$ and call F the limit in the mean of order p'. A similar definition is understood when n is replaced by a continuously varying parameter.

Let F and G be defined on $C[0,t]$ and let X_{n+1} be given by (1.3). For $\lambda \in \mathbb{C}_+$ and w_φ-a.e. $y \in C[0,t]$, let

$$T_\lambda[F|X_{n+1}](y, \vec{\xi}_{n+1}) = E^{anw_\lambda}[F(y+\cdot)|X_{n+1}](\vec{\xi}_{n+1})$$

for $P_{X_{n+1}}$-a.e. $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$ if it exists. For a nonzero real q and w_φ-a.e. $y \in C[0,t]$, define the L_1-analytic conditional Fourier-Feynman transform $T_q^{(1)}[F|X_{n+1}]$ of F given X_{n+1} by the formula

$$T_q^{(1)}[F|X_{n+1}](y, \vec{\xi}_{n+1}) = E^{anf_q}[F(y+\cdot)|X_{n+1}](\vec{\xi}_{n+1})$$

for $P_{X_{n+1}}$-a.e. $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$ if it exists. For $1 < p \le \infty$, define the L_p- analytic conditional Fourier-Feynman transform $T_q^{(p)}[F|X_{n+1}]$ of F given X_{n+1} by the formula

$$T_q^{(p)}[F|X_{n+1}](\cdot, \vec{\xi}_{n+1}) = \underset{\lambda\to -iq}{\text{l.i.m.}}(w^{p'})(T_\lambda[F|X_{n+1}](\cdot, \vec{\xi}_{n+1}))$$

for $P_{X_{n+1}}$-a.e. $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$, where λ approaches to $-iq$ through $\mathbb{C}_+$. We also define the conditional convolution product $[(F * G)_\lambda|X_{n+1}]$ of F and G given X_{n+1} by the formula, for w_φ-a.e. $y \in C[0,t]$

$$
\begin{aligned}
&[(F * G)_\lambda | X_{n+1}](y, \vec{\xi}_{n+1}) \\
&= \begin{cases} E^{anw_\lambda}\left[F\left(\dfrac{y+\cdot}{\sqrt{2}}\right)G\left(\dfrac{y-\cdot}{\sqrt{2}}\right)\middle| X_{n+1}\right](\vec{\xi}_{n+1}), & \lambda \in \mathbb{C}_+; \\ E^{anf_q}\left[F\left(\dfrac{y+\cdot}{\sqrt{2}}\right)G\left(\dfrac{y-\cdot}{\sqrt{2}}\right)\middle| X_{n+1}\right](\vec{\xi}_{n+1}), & \lambda = -iq \end{cases}
\end{aligned}
$$

if they exist for $P_{X_{n+1}}$-a.e. $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$. If $\lambda = -iq$, we replace $[(F * G)_\lambda | X_{n+1}]$ by $[(F * G)_q | X_{n+1}]$. Similar definitions and notations are understood with $\vec{\xi}_n \in \mathbb{R}^{n+1}$ if X_{n+1} is replaced by X_n which is given by (1.4).

2. Conditional Fourier-Feynman transform with final time conditioning function

For v in $L_2[0,t]$ and x in $C[0,t]$, let (v,x) denote the Paley-Wiener-Zygmund integral of v according to x ([11]). Note that $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle_{\mathbb{R}^r}$ denote the inner product over $L_2[0,t]$ and the dot product on the r-dimensional Euclidean space $\mathbb{R}^r$, respectively.

For each $j = 1, \cdots, n+1$, let $\alpha_j = (t_j - t_{j-1})^{-\frac{1}{2}}\chi_{(t_{j-1}, t_j]}$ on $[0,t]$. Let V be the subspace of $L_2[0,t]$ generated by $\{\alpha_1, \cdots, \alpha_{n+1}\}$ and $V^\perp$ denote the orthogonal complement of V. Let $\mathcal{P}$ and $\mathcal{P}^\perp$ be the orthogonal projections from $L_2[0,t]$ to V and $V^\perp$, respectively. Throughout this paper, let $\{v_1, v_2, \cdots, v_r\}$ be an orthonormal subset of $L_2[0,t]$ such that $\{\mathcal{P}^\perp v_1, \cdots, \mathcal{P}^\perp v_r\}$ is an independent set. Let $\{e_1, \cdots, e_r\}$ be the orthonormal set obtained from $\{\mathcal{P}^\perp v_1, \cdots, \mathcal{P}^\perp v_r\}$ by the Gram-Schmidt orthonormalization process. Now, for $l = 1, \cdots, r$, let $\mathcal{P}^\perp v_l = \sum_{j=1}^r \beta_{lj} e_j$ be the linear combinations of the e_js and let $A = [\beta_{lj}]_{r\times r}$ be the coefficient matrix of the combinations. We can regard A as the linear transformation $T_A : \mathbb{R}^r \to \mathbb{R}^r$ given by $T_A \vec{z} = \vec{z}A$, where $\vec{z}$ is any row-vector in $\mathbb{R}^r$. Note that A is invertible so that T_A is an isomorphism. For $v \in L_2[0,t]$, let

$$c_j(v) = \langle v, e_j \rangle \tag{2.1}$$

for $j = 1, \cdots, r$ and let

$$(\vec{v}, x) = ((v_1, x), \cdots, (v_r, x)) \tag{2.2}$$

for $x \in C[0,t]$. Furthermore, for $\vec{z} \in \mathbb{R}^r$ and $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$ let

$$H_1(x, \vec{\xi}_{n+1}, v, \vec{z}) = \exp\{i[(v, x + [\vec{\xi}_{n+1}]) + \langle (\vec{v}, x + [\vec{\xi}_{n+1}]), \vec{z} \rangle_{\mathbb{R}^r}]\} \tag{2.3}$$

and for $\lambda \in \mathbb{C}_+^\sim$ let

$$H_2(\lambda, v, \vec{z}) = \exp\Big\{-\frac{1}{2\lambda}[\|\mathcal{P}^\perp v\|_2^2 + 2\langle \vec{c}(\mathcal{P}^\perp v), T_A\vec{z}\rangle_{\mathbb{R}^r} + \|T_A\vec{z}\|_{\mathbb{R}^r}^2]\Big\}, \tag{2.4}$$

where $\vec{c}(\mathcal{P}^\perp v) = (c_1(\mathcal{P}^\perp v), \cdots, c_r(\mathcal{P}^\perp v))$ and the c_js are given by (2.1). Note that by the Bessel's inequality,

$$\begin{aligned} &|H_2(\lambda, v, \vec{z})| \\ &= \exp\Big\{-\frac{\mathrm{Re}\lambda}{2|\lambda|^2}[\|\mathcal{P}^\perp v\|_2^2 - \|\vec{c}(\mathcal{P}^\perp v)\|_{\mathbb{R}^r}^2 + \|\vec{c}(\mathcal{P}^\perp v) + T_A\vec{z}\|_{\mathbb{R}^r}^2]\Big\} \\ &\le 1. \end{aligned} \tag{2.5}$$

Using the same method as used in the proof of Theorem 2.6 in [10], we can prove the following lemma.

LEMMA 2.1. *For $x \in C[0,t]$, $\lambda > 0$, $v \in L_2[0,t]$ and $\vec{z} \in \mathbb{R}^r$, let*

$$H_3(\lambda, v, \vec{z}, x) = \exp\{i\lambda^{-\frac{1}{2}}[(v, x - [x]) + \langle(\vec{v}, x - [x]), \vec{z}\rangle_{\mathbb{R}^r}]\}. \tag{2.6}$$

Then

$$\int_C H_3(\lambda, v, \vec{z}, x)\,dw_\varphi(x) = H_2(\lambda, v, \vec{z}),$$

where H_2 is given by (2.4).

Let $\hat{\mathrm{M}}(\mathbb{R}^r)$ be the set of all functions ϕ on $\mathbb{R}^r$ defined by

$$\phi(\vec{u}) = \int_{\mathbb{R}^r} \exp\{i\langle \vec{u}, \vec{z}\rangle_{\mathbb{R}^r}\}\,d\rho(\vec{z}), \tag{2.7}$$

where ρ is a complex Borel measure of bounded variation over $\mathbb{R}^r$, and let Φ be given by

$$\Phi(x) = \phi(\vec{v}, x) \tag{2.8}$$

for w_φ-a.e. $x \in C[0,t]$, where $(\vec{v}, x)$ and $\phi \in \hat{\mathrm{M}}(\mathbb{R}^r)$ are given by (2.2) and (2.7), respectively. Let $\mathcal{M} = \mathcal{M}(L_2[0,t])$ be the class of all $\mathbb{C}$-valued Borel measures of bounded variation over $L_2[0,t]$ and let $\mathcal{S}_{w_\varphi}$ be the space of all functions F which for $\sigma \in \mathcal{M}$ have the form

$$F(x) = \int_{L_2[0,t]} \exp\{i(v, x)\}\,d\sigma(v) \tag{2.9}$$

for w_φ-a.e. $x \in C[0,t]$. Note that $\mathcal{S}_{w_\varphi}$ is a Banach algebra which is equivalent to $\mathcal{M}$ with the norm $\|F\| = \|\sigma\|$, the total variation of σ [11].

Now we have the following theorem.

THEOREM 2.2. *Let $1 \le p \le \infty$ and X_{n+1} be given by (1.3). For w_φ-a.e. $x \in C[0,t]$, let $\Psi(x) = F(x)\Phi(x)$, where Φ and F are given by (2.8) and (2.9), respectively. Then for a nonzero real q, w_φ-a.e. $y \in C[0,t]$ and $P_{X_{n+1}}$-a.e. $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$,*

$$\begin{aligned}(2.10)\qquad & T_q^{(p)}[\Psi|X_{n+1}](y,\vec{\xi}_{n+1}) \\ &= \int_{\mathbb{R}^r}\int_{L_2[0,t]} H_1(y,\vec{\xi}_{n+1},v,\vec{z})H_2(-iq,v,\vec{z})d\sigma(v)d\rho(\vec{z}),\end{aligned}$$

where H_1 and H_2 are given by (2.3) and (2.4), respectively.

Proof. For $\lambda > 0$, $y \in C[0,t]$ and $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$,

$$\begin{aligned}& I_\Psi^\lambda(y,\vec{\xi}_{n+1}) \\ &= \int_{\mathbb{R}^r}\int_{L_2[0,t]}\int_C H_1(y,\vec{\xi}_{n+1},v,\vec{z})H_3(\lambda,v,\vec{z},x)dw_\varphi(x)d\sigma(v)d\rho(\vec{z}),\end{aligned}$$

where H_3 is given by (2.6). By Lemma 2.1, we obtain that

$$I_\Psi^\lambda(y,\vec{\xi}_{n+1}) = \int_{\mathbb{R}^r}\int_{L_2[0,t]} H_1(y,\vec{\xi}_{n+1},v,\vec{z})H_2(\lambda,v,\vec{z})d\sigma(v)d\rho(\vec{z}).$$

By (2.5), the Moreras theorem and the dominated convergence theorem, we have the existence of $T_\lambda[\Psi|X_{n+1}](y,\vec{\xi}_{n+1})$ as the analytic extension of $I_\Psi^\lambda(y,\vec{\xi}_{n+1})$ on $\mathbb{C}_+$. Let $T_q^{(p)}[\Psi|X_{n+1}](y,\vec{\xi}_{n+1})$ be given by (2.10) and $\frac{1}{p}+\frac{1}{p'}=1$. Then

$$\begin{aligned}& \|T_\lambda[\Psi|X_{n+1}](\cdot,\vec{\xi}_{n+1}) - T_q^{(p)}[\Psi|X_{n+1}](\cdot,\vec{\xi}_{n+1})\|_{p'} \\ &\le \int_{\mathbb{R}^r}\int_{L_2[0,t]} |H_2(\lambda,v,\vec{z}) - H_2(-iq,v,\vec{z})|d|\sigma|(v)d|\rho|(\vec{z})\end{aligned}$$

which converges to 0 as λ approaches to $-iq$ through $\mathbb{C}_+$ by the dominated convergence theorem. Now, the proof is completed. □

THEOREM 2.3. *Let ϕ_1, ϕ_2 and ρ_1, ρ_2 be related by (2.7), respectively, and let F_1, F_2 and σ_1, σ_2 be related by (2.9), respectively. Let $\Psi_1(x) = F_1(x)\phi_1(\vec{v},x)$ and $\Psi_2(x) = F_2(x)\phi_2(\vec{v},x)$ for w_φ-a.e. $x \in C[0,t]$. Furthermore, let X_{n+1} be given by (1.3). Then for a nonzero real q, w_φ-a.e.*

$y \in C[0,t]$ and $P_{X_{n+1}}$-a.e. $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$,

$$[(\Psi_1 * \Psi_2)_q | X_{n+1}](y, \vec{\xi}_{n+1})$$
$$= \int_{\mathbb{R}^r} \int_{\mathbb{R}^r} \int_{L_2[0,t]} \int_{L_2[0,t]} H_1 \left(y, \vec{\xi}_{n+1}, \frac{1}{\sqrt{2}} v_1, \frac{1}{\sqrt{2}} \vec{z}_1 \right) H_1 \Big(y,$$
$$-\vec{\xi}_{n+1}, \frac{1}{\sqrt{2}} v_2, \frac{1}{\sqrt{2}} \vec{z}_2 \Big) H_2 \left(-iq, \frac{1}{\sqrt{2}} (v_1 - v_2), \frac{1}{\sqrt{2}} (\vec{z}_1 - \vec{z}_2) \right)$$
$$d\sigma_1(v_1) d\sigma_2(v_2) d\rho_1(\vec{z}_1) d\rho_2(\vec{z}_2),$$

where H_1 and H_2 are given by (2.3) and (2.4), respectively.

Proof. For $\lambda > 0$, w_φ-a.e. $y \in C[0,t]$ and $P_{X_{n+1}}$-a.e. $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$,

$$[(\Psi_1 * \Psi_2)_\lambda | X_{n+1}](y, \vec{\xi}_{n+1})$$
$$= \int_{\mathbb{R}^r} \int_{\mathbb{R}^r} \int_{L_2[0,t]} \int_{L_2[0,t]} \int_C \exp \Big\{ \frac{i}{\sqrt{2}} [(v_1, y + [\vec{\xi}_{n+1}]) + \langle (\vec{v}, y +$$
$$[\vec{\xi}_{n+1}]), \vec{z}_1 \rangle_{\mathbb{R}^r} + (v_2, y - [\vec{\xi}_{n+1}]) + \langle (\vec{v}, y - [\vec{\xi}_{n+1}]), \vec{z}_2 \rangle_{\mathbb{R}^r}] +$$
$$\frac{i}{\sqrt{2\lambda}} [(v_1 - v_2, x - [x]) + \langle (\vec{v}, x - [x]), \vec{z}_1 - \vec{z}_2 \rangle_{\mathbb{R}^r}] \Big\} dw_\varphi(x)$$
$$d\sigma_1(v_1) d\sigma_2(v_2) d\rho_1(\vec{z}_1) d\rho_2(\vec{z}_2)$$
$$= \int_{\mathbb{R}^r} \int_{\mathbb{R}^r} \int_{L_2[0,t]} \int_{L_2[0,t]} \int_C H_1 \left(y, \vec{\xi}_{n+1}, \frac{1}{\sqrt{2}} v_1, \frac{1}{\sqrt{2}} \vec{z}_1 \right) H_1 \Big(y,$$
$$-\vec{\xi}_{n+1}, \frac{1}{\sqrt{2}} v_2, \frac{1}{\sqrt{2}} \vec{z}_2 \Big) H_3 \left(\lambda, \frac{1}{\sqrt{2}} (v_1 - v_2), \frac{1}{\sqrt{2}} (\vec{z}_1 - \vec{z}_2), x \right)$$
$$dw_\varphi(x) d\sigma_1(v_1) d\sigma_2(v_2) d\rho_1(\vec{z}_1) d\rho_2(\vec{z}_2),$$

where H_3 is given by (2.6). By Lemma 2.1, we obtain that

$$[(\Psi_1 * \Psi_2)_\lambda | X_{n+1}](y, \vec{\xi}_{n+1})$$
$$= \int_{\mathbb{R}^r} \int_{\mathbb{R}^r} \int_{L_2[0,t]} \int_{L_2[0,t]} H_1 \left(y, \vec{\xi}_{n+1}, \frac{1}{\sqrt{2}} v_1, \frac{1}{\sqrt{2}} \vec{z}_1 \right) H_1 \Big(y,$$
$$-\vec{\xi}_{n+1}, \frac{1}{\sqrt{2}} v_2, \frac{1}{\sqrt{2}} \vec{z}_2 \Big) H_2 \left(\lambda, \frac{1}{\sqrt{2}} (v_1 - v_2), \frac{1}{\sqrt{2}} (\vec{z}_1 - \vec{z}_2) \right)$$
$$d\sigma_1(v_1) d\sigma_2(v_2) d\rho_1(\vec{z}_1) d\rho_2(\vec{z}_2).$$

By (2.5), the Morera's theorem and the dominated convergence theorem, we have the result. $\square$

3. Conditional Fourier-Feynman transform without final time conditioning function

In this section, we evaluate time-independent conditional Fourier-Feynman transform and conditional convolution product of the functions as given in the previous section.

LEMMA 3.1. *For* $\vec{\xi}_n = (\xi_0, \xi_1, \cdots, \xi_n) \in \mathbb{R}^{n+1}$, *let* $\vec{\xi}_{n+1} = (\xi_0, \xi_1, \cdots, \xi_n, \xi_{n+1})$, *where* $\xi_{n+1} \in \mathbb{R}$. *Furthermore, for* $y \in C[0,t]$, $v \in L_2[0,t]$, $\vec{z} = (z_1, \cdots, z_r) \in \mathbb{R}^r$, *let* $H_1(y, \vec{\xi}_{n+1}, v, \vec{z})$ *be given by (2.3), let*

$$(3.1)\quad H_4(\vec{\xi}_n, v, \vec{z}) = \exp\bigg\{ i\sum_{j=1}^{n}(\xi_j - \xi_{j-1})\bigg[(\mathcal{P}v)(t_j) + \sum_{l=1}^{r} z_l(\mathcal{P}v_l)(t_j)\bigg]\bigg\}$$

and for $\lambda \in \mathbb{C}_+^{\sim}$, *let*

$$(3.2)\quad H_5(\lambda, v, \vec{z}) = \exp\bigg\{-\frac{1}{2\lambda}\bigg[(\mathcal{P}v)(t) + \sum_{l=1}^{r} z_l(\mathcal{P}v_l)(t)\bigg]^2\bigg\}.$$

Then for $\lambda > 0$,

$$(3.3)\quad \bigg[\frac{\lambda}{2\pi(t-t_n)}\bigg]^{\frac{1}{2}} \int_{\mathbb{R}} H_1(y, \vec{\xi}_{n+1}, v, \vec{z}) \exp\bigg\{-\frac{\lambda(\xi_{n+1}-\xi_n)^2}{2(t-t_n)}\bigg\} d\xi_{n+1}$$
$$= H_1(y, 0, v, \vec{z}) H_4(\vec{\xi}_n, v, \vec{z}) H_5(\lambda, v, \vec{z}).$$

Proof. By the definition of the Paley-Wiener-Zygmund integral, it is not difficult to show that for $v \in L_2[0,t]$

$$(v, [\vec{\xi}_{n+1}]) = \sum_{j=1}^{n}(\mathcal{P}v)(t_j)(\xi_j - \xi_{j-1}) + (\mathcal{P}v)(t)(\xi_{n+1} - \xi_n).$$

Now, let I_λ be the left hand side of (3.3). Then

$$\begin{aligned} I_\lambda &= H_1(y, 0, v, \vec{z}) \bigg[\frac{\lambda}{2\pi(t-t_n)}\bigg]^{\frac{1}{2}} \int_{\mathbb{R}} H_4(\vec{\xi}_n, v, \vec{z}) \exp\bigg\{ i\bigg[(\mathcal{P}v)(t) \\ &\quad + \sum_{l=1}^{r} z_l(\mathcal{P}v_l)(t)\bigg](\xi_{n+1} - \xi_n) - \frac{\lambda(\xi_{n+1}-\xi_n)^2}{2(t-t_n)}\bigg\} d\xi_{n+1} \\ &= H_1(y, 0, v, \vec{z}) H_4(\vec{\xi}_n, v, \vec{z}) H_5(\lambda, v, \vec{z}) \end{aligned}$$

by the following well-known integration formula

$$\int_{\mathbb{R}} \exp\{-au^2 + ibu\} du = \Big(\frac{\pi}{a}\Big)^{\frac{1}{2}} \exp\Big\{-\frac{b^2}{4a}\Big\}$$

for $a \in \mathbb{C}_+$ and $b \in \mathbb{R}$. Now, the proof is completed. $\square$

THEOREM 3.2. *Let $1 \le p \le \infty$ and X_n be given by (1.4). For w_φ-a.e. $x \in C[0,t]$, let $\Psi(x) = F(x)\Phi(x)$, where Φ and F are given by (2.8) and (2.9), respectively. Then for a nonzero real q, w_φ-a.e. $y \in C[0,t]$ and P_{X_n}-a.e. $\vec{\xi}_n \in \mathbb{R}^{n+1}$,*

$$\begin{aligned} T_q^{(p)}[\Psi|X_n](y,\vec{\xi}_n) &= \int_{\mathbb{R}^r}\int_{L_2[0,t]} H_1(y,0,v,\vec{z})H_4(\vec{\xi}_n,v,\vec{z}) \\ &\quad \times H_2(-iq,v,\vec{z})H_5(-iq,v,\vec{z})d\sigma(v)d\rho(\vec{z}), \end{aligned}$$

where H_1, H_2, H_4, and H_5 are given by (2.3), (2.4), (3.1) and (3.2), respectively.

Proof. For $\vec{\xi}_n = (\xi_0, \xi_1, \cdots, \xi_n) \in \mathbb{R}^{n+1}$, let $\vec{\xi}_{n+1} = (\xi_0, \xi_1, \cdots, \xi_n, \xi_{n+1})$, where $\xi_{n+1} \in \mathbb{R}$. For $\lambda > 0$, $y \in C[0,t]$ and $\vec{\xi}_n \in \mathbb{R}^{n+1}$,

$$\begin{aligned} K_\Psi^\lambda(y,\vec{\xi}_n) &= \left[\frac{\lambda}{2\pi(t-t_n)}\right]^{\frac{1}{2}} \int_{\mathbb{R}} I_\Psi^\lambda(y,\vec{\xi}_{n+1}) \exp\left\{-\frac{\lambda(\xi_{n+1}-\xi_n)^2}{2(t-t_n)}\right\} d\xi_{n+1} \\ &= \left[\frac{\lambda}{2\pi(t-t_n)}\right]^{\frac{1}{2}} \int_{\mathbb{R}^r}\int_{L_2[0,t]} H_2(\lambda,v,\vec{z}) \int_{\mathbb{R}} H_1(y,\vec{\xi}_{n+1},v,\vec{z}) \\ &\quad \times \exp\left\{-\frac{\lambda(\xi_{n+1}-\xi_n)^2}{2(t-t_n)}\right\} d\xi_{n+1} d\sigma(v) d\rho(\vec{z}) \end{aligned}$$

by Theorem 2.2. By Lemma 3.1, we obtain that

$$\begin{aligned} K_\Psi^\lambda(y,\vec{\xi}_n) &= \int_{\mathbb{R}^r}\int_{L_2[0,t]} H_1(y,0,v,\vec{z})H_2(\lambda,v,\vec{z})H_4(\vec{\xi}_n,v,\vec{z}) \\ &\quad \times H_5(\lambda,v,\vec{z})d\sigma(v)d\rho(\vec{z}). \end{aligned}$$

By (2.5), the Morera's theorem and the dominated convergence theorem, we have the existence of $T_\lambda[\Psi|X_n](y,\vec{\xi}_n)$ as the analytic extension of $K_\Psi^\lambda(y,\vec{\xi}_n)$ on $\mathbb{C}_+$. Let $\frac{1}{p} + \frac{1}{p'} = 1$. Then for $\lambda \in \mathbb{C}_+$,

$$\begin{aligned} &\|T_\lambda[\Psi|X_n](\cdot,\vec{\xi}_n) - T_q^{(p)}[\Psi|X_n](\cdot,\vec{\xi}_n)\|_{p'} \\ &\le \int_{\mathbb{R}^r}\int_{L_2[0,t]} |H_2(\lambda,v,\vec{z})H_5(\lambda,v,\vec{z}) - H_2(-iq,v,\vec{z})H_5(-iq,v,\vec{z})| d|\sigma|(v) d|\rho|(\vec{z}) \end{aligned}$$

which converges to 0 as λ approaches to $-iq$ through $\mathbb{C}_+$. Now, the proof is completed. $\square$

THEOREM 3.3. *Let Ψ_1 and Ψ_2 be as given in Theorem 2.3, and let X_n be given by (1.4). Then for a nonzero real q, w_φ-a.e. $y \in C[0,t]$ and P_{X_n}-a.e. $\vec{\xi}_n \in \mathbb{R}^{n+1}$, $[(\Psi_1 * \Psi_2)_q|X_n](y,\vec{\xi}_n)$ is given by*

$$\begin{aligned}
&[(\Psi_1 * \Psi_2)_q|X_n](y,\vec{\xi}_n)\\
&= \int_{\mathbb{R}^r}\int_{\mathbb{R}^r}\int_{L_2[0,t]}\int_{L_2[0,t]} H_1\Big(y,0,\frac{1}{\sqrt{2}}(v_1+v_2),\frac{1}{\sqrt{2}}(\vec{z}_1+\vec{z}_2)\Big)H_4\Big(\vec{\xi}_n,\\
&\frac{1}{\sqrt{2}}(v_1-v_2),\frac{1}{\sqrt{2}}(\vec{z}_1-\vec{z}_2)\Big)H_2\Big(-iq,\frac{1}{\sqrt{2}}(v_1-v_2),\frac{1}{\sqrt{2}}(\vec{z}_1-\vec{z}_2)\Big)\\
&\times H_5\Big(-iq,\frac{1}{\sqrt{2}}(v_1-v_2),\frac{1}{\sqrt{2}}(\vec{z}_1-\vec{z}_2)\Big)d\sigma_1(v_1)d\sigma_2(v_2)d\rho_1(\vec{z}_1)d\rho_2(\vec{z}_2),
\end{aligned}$$

where H_1, H_2, H_4 and H_5 are given by (2.3), (2.4), (3.1) and (3.2), respectively.

Proof. For $\vec{\xi}_n = (\xi_0,\xi_1,\cdots,\xi_n) \in \mathbb{R}^{n+1}$, let $\vec{\xi}_{n+1} = (\xi_0,\xi_1,\cdots,\xi_n,\xi_{n+1})$, where $\xi_{n+1} \in \mathbb{R}$. Note that for $y \in C[0,t]$, for $v_1, v_2 \in L_2[0,t]$ and for $\vec{z}_1,\vec{z}_2 \in \mathbb{R}^r$

$$\begin{aligned}
&H_1\left(y,\vec{\xi}_{n+1},\frac{1}{\sqrt{2}}v_1,\frac{1}{\sqrt{2}}\vec{z}_1\right)H_1\left(y,-\vec{\xi}_{n+1},\frac{1}{\sqrt{2}}v_2,\frac{1}{\sqrt{2}}\vec{z}_2\right)\\
&= H_1\Big(y,0,\frac{1}{\sqrt{2}}(v_1+v_2),\frac{1}{\sqrt{2}}(\vec{z}_1+\vec{z}_2)\Big)\\
&\qquad\times H_1\Big(0,\vec{\xi}_{n+1},\frac{1}{\sqrt{2}}(v_1-v_2),\frac{1}{\sqrt{2}}(\vec{z}_1-\vec{z}_2)\Big)
\end{aligned}$$

so that we have for $\lambda > 0$

$$\begin{aligned}
I_\lambda(\vec{\xi}_n) \;\equiv\;& \left[\frac{\lambda}{2\pi(t-t_n)}\right]^{\frac{1}{2}}\int_{\mathbb{R}} H_1\left(y,\vec{\xi}_{n+1},\frac{1}{\sqrt{2}}v_1,\frac{1}{\sqrt{2}}\vec{z}_1\right)H_1\Big(y,-\vec{\xi}_{n+1},\\
&\frac{1}{\sqrt{2}}v_2,\frac{1}{\sqrt{2}}\vec{z}_2\Big)\exp\Big\{-\frac{\lambda(\xi_{n+1}-\xi_n)^2}{2(t-t_n)}\Big\}d\xi_{n+1}\\
=\;& H_1\Big(y,0,\frac{1}{\sqrt{2}}(v_1+v_2),\frac{1}{\sqrt{2}}(\vec{z}_1+\vec{z}_2)\Big)H_4\Big(\vec{\xi}_n,\frac{1}{\sqrt{2}}(v_1-v_2),\\
&\frac{1}{\sqrt{2}}(\vec{z}_1-\vec{z}_2)\Big)H_5\Big(\lambda,\frac{1}{\sqrt{2}}(v_1-v_2),\frac{1}{\sqrt{2}}(\vec{z}_1-\vec{z}_2)\Big)
\end{aligned}$$

by Lemma 3.1. Now, by Theorem 2.3, we obtain that

$$
\begin{aligned}
&[(\Psi_1 * \Psi_2)_\lambda | X_n](y, \vec{\xi}_n) \\
&= \int_{\mathbb{R}^r} \int_{\mathbb{R}^r} \int_{L_2[0,t]} \int_{L_2[0,t]} H_1\bigg(y, 0, \frac{1}{\sqrt{2}}(v_1 + v_2), \frac{1}{\sqrt{2}}(\vec{z}_1 + \vec{z}_2)\bigg) H_2\bigg(\lambda, \\
&\frac{1}{\sqrt{2}}(v_1 - v_2), \frac{1}{\sqrt{2}}(\vec{z}_1 - \vec{z}_2)\bigg) H_4\bigg(\vec{\xi}_n, \frac{1}{\sqrt{2}}(v_1 - v_2), \frac{1}{\sqrt{2}}(\vec{z}_1 - \vec{z}_2)\bigg) \\
&\times H_5\bigg(\lambda, \frac{1}{\sqrt{2}}(v_1 - v_2), \frac{1}{\sqrt{2}}(\vec{z}_1 - \vec{z}_2)\bigg) d\sigma_1(v_1) d\sigma_2(v_2) d\rho_1(\vec{z}_1) d\rho_2(\vec{z}_2).
\end{aligned}
$$

By (2.5), the Morera's theorem and the dominated convergence theorem, we have the result. □

4. Relationships between conditional Fourier-Feynman transforms and convolutions

In this section, we investigate the inverse conditional transform of the conditional Fourier-Feynman transforms of the functions as given in the previous sections.

THEOREM 4.1. *Let $1 \le p \le \infty$. Then, under the assumptions as given in Theorem 2.2, we have for $P_{X_{n+1}}$-a.e. $\vec{\xi}_{n+1}, \vec{\zeta}_{n+1} \in \mathbb{R}^{n+2}$*

$$
\| T_{\bar{\lambda}}[T_\lambda[\Psi | X_{n+1}](\cdot, \vec{\xi}_{n+1}) | X_{n+1}](\cdot, \vec{\zeta}_{n+1}) - \Psi(\cdot + [\vec{\zeta}_{n+1} + \vec{\xi}_{n+1}]) \|_p \to 0
$$

as λ approaches to $-iq$ through $\mathbb{C}_+$.

Proof. For $\lambda_1 > 0$, $\lambda \in \mathbb{C}_+$, w_φ-a.e. $y \in C[0,t]$ and $P_{X_{n+1}}$-a.e. $\vec{\xi}_{n+1}, \vec{\zeta}_{n+1} \in \mathbb{R}^{n+2}$,

$$
\begin{aligned}
&I^{\lambda_1}_{T_\lambda[\Psi|X_{n+1}](\cdot, \vec{\xi}_{n+1})}(y, \vec{\zeta}_{n+1}) \\
&= \int_{\mathbb{R}^r} \int_{L_2[0,t]} \int_C H_1(\lambda_1^{-\frac{1}{2}}(x - [x]) + y + [\vec{\zeta}_{n+1}], \vec{\xi}_{n+1}, v, \vec{z}) H_2(\lambda, v, \vec{z}) \\
&\qquad dw_\varphi(x) d\sigma(v) d\rho(\vec{z}) \\
&= \int_{\mathbb{R}^r} \int_{L_2[0,t]} \int_C H_1(y, \vec{\zeta}_{n+1} + \vec{\xi}_{n+1}, v, \vec{z}) H_2(\lambda.v.\vec{z}) H_3(\lambda_1, v, \vec{z}, x) \\
&\qquad dw_\varphi(x) d\sigma(v) d\rho(\vec{z}) \\
&= \int_{\mathbb{R}^r} \int_{L_2[0,t]} H_1(y, \vec{\zeta}_{n+1} + \vec{\xi}_{n+1}, v, \vec{z}) H_2(\lambda_1, v, \vec{z}) H_2(\lambda, v, \vec{z}) d\sigma(v) d\rho(\vec{z})
\end{aligned}
$$

by Lemma 2.1 and Theorem 2.2, where H_1, H_2 and H_3 are given by (2.3), (2.4) and (2.6), respectively. By (2.5) and the Morera's Theorem, we have for $\lambda_1 \in \mathbb{C}_+$

$$\begin{aligned}&T_{\lambda_1}[T_\lambda[\Psi|X_{n+1}](\cdot,\vec{\xi}_{n+1})|X_{n+1}](y,\vec{\zeta}_{n+1})\\&=\int_{\mathbb{R}^r}\int_{L_2[0,t]} H_1(y,\vec{\zeta}_{n+1}+\vec{\xi}_{n+1},v,\vec{z})H_2(\lambda_1,v,\vec{z})H_2(\lambda,v,\vec{z})d\sigma(v)d\rho(\vec{z}).\end{aligned}$$

Now, we have for $\lambda \in \mathbb{C}_+$

$$\begin{aligned}&T_{\overline{\lambda}}[T_\lambda[\Psi|X_{n+1}](\cdot,\vec{\xi}_{n+1})|X_{n+1}](y,\vec{\zeta}_{n+1})\\&=\int_{\mathbb{R}^r}\int_{L_2[0,t]} H_1(y,\vec{\zeta}_{n+1}+\vec{\xi}_{n+1},v,\vec{z})H_2\left(\frac{|\lambda|^2}{2\mathrm{Re}\lambda},v,\vec{z}\right)d\sigma(v)d\rho(\vec{z})\end{aligned}$$

so that

$$\begin{aligned}&\|T_{\overline{\lambda}}[T_\lambda[\Psi|X_{n+1}](\cdot,\vec{\xi}_{n+1})|X_{n+1}](\cdot,\vec{\zeta}_{n+1})-\Psi(\cdot+[\vec{\zeta}_{n+1}+\vec{\xi}_{n+1}])\|_p\\&\le\int_{\mathbb{R}^r}\int_{L_2[0,t]}\left[1-H_2\left(\frac{|\lambda|^2}{2\mathrm{Re}\lambda},v,\vec{z}\right)\right]d|\sigma|(v)d|\rho|(\vec{z})\end{aligned}$$

which converges to 0 as λ approaches to $-iq$ through $\mathbb{C}_+$ by the dominated convergence theorem. □

THEOREM 4.2. *Let $1 \le p \le \infty$ and X_n be given by (1.4). For w_φ-a.e. $x \in C[0,t]$, let $\Psi(x) = F(x)\Phi(x)$, where Φ and F are given by (2.8) and (2.9), respectively. Furthermore, for $y \in C[0,t]$ and $\vec{u}_n \in \mathbb{R}^{n+1}$ let*

$$\begin{aligned}&\Psi_1(y,\vec{u}_n)\\&=\int_{\mathbb{R}^r}\int_{L_2[0,t]}\exp\{i(v,y)\}\exp\{i\langle(\vec{v},y),\vec{z}\rangle_{\mathbb{R}^r}\}H_4(\vec{u}_n,v,\vec{z})d\sigma(v)d\rho(\vec{z}),\end{aligned}$$

where H_4 is given by (3.1). Then, for P_{X_n}-a.e. $\vec{\xi}_n,\vec{\zeta}_n \in \mathbb{R}^{n+1}$

$$\|T_{\overline{\lambda}}[T_\lambda[\Psi|X_n](\cdot,\vec{\xi}_n)|X_n](\cdot,\vec{\zeta}_n)-\Psi_1(\cdot,\vec{\zeta}_n+\vec{\xi}_n)\|_p \to 0$$

as λ approaches to $-iq$ through $\mathbb{C}_+$.

Proof. For $\vec{\zeta}_n = (\zeta_0,\zeta_1,\cdots,\zeta_n) \in \mathbb{R}$, let $\vec{\zeta}_{n+1} = (\zeta_0,\zeta_1,\cdots,\zeta_n,\zeta_{n+1})$, where $\zeta_{n+1} \in \mathbb{R}$. Then, for $\lambda_1 > 0$ and $\lambda \in \mathbb{C}_+$

$$\begin{aligned}I^{\lambda_1}_{T_\lambda[\Psi|X_n](\cdot,\vec{\xi}_n)}(y,\vec{\zeta}_{n+1}) &= \int_{\mathbb{R}^r}\int_{L_2[0,t]} H_1(y,\vec{\zeta}_{n+1},v,\vec{z})H_2(\lambda_1,v,\vec{z})\\&\quad\times H_2(\lambda,v,\vec{z})H_4(\vec{\xi}_n,v,\vec{z})H_5(\lambda,v,\vec{z})d\sigma(v)d\rho(\vec{z})\end{aligned}$$

by Theorems 2.2 and 3.2, so that by Lemma 3.1

$$
\begin{aligned}
&K^{\lambda_1}_{T_\lambda[\Psi|X_n](\cdot,\vec{\xi}_n)}(y,\vec{\zeta}_n)\\
&=\left[\frac{\lambda}{2\pi(t-t_n)}\right]^{\frac{1}{2}}\int_{\mathbb{R}} I^{\lambda_1}_{T_\lambda[\Psi|X_n](\cdot,\vec{\xi}_n)}(y,\vec{\zeta}_{n+1})\exp\left\{-\frac{\lambda(\zeta_{n+1}-\zeta_n)^2}{2(t-t_n)}\right\}d\zeta_{n+1}\\
&=\int_{\mathbb{R}^r}\int_{L_2[0,t]} H_1(y,0,v,\vec{z})H_2(\lambda_1,v,\vec{z})H_2(\lambda,v,\vec{z})H_4(\vec{\zeta}_n+\vec{\xi}_n,v,\vec{z})\\
&\qquad\times H_5(\lambda_1,v,\vec{z})H_5(\lambda,v,\vec{z})d\sigma(v)d\rho(\vec{z}),
\end{aligned}
$$

where H_1, H_2 and H_5 are given by (2.3), (2.4) and (3.2), respectively. By (2.5), the Moreras theorem and the dominated convergence theorem, we have the existence of $T_{\lambda_1}[T_\lambda[\Psi|X_n](\cdot,\vec{\xi}_n)|X_n](y,\vec{\zeta}_n)$ with respect to $\lambda_1\in\mathbb{C}_+$ as the analytic extension of $K^{\lambda_1}_{T_\lambda[\Psi|X_n](\cdot,\vec{\xi}_n)}(y,\vec{\zeta}_n)$. Now, for $\lambda\in\mathbb{C}_+$ and $y\in C[0,t]$

$$
\begin{aligned}
&T_{\overline{\lambda}}[T_\lambda[\Psi|X_n](\cdot,\vec{\xi}_n)|X_n](y,\vec{\zeta}_n)\\
&=\int_{\mathbb{R}^r}\int_{L_2[0,t]} H_1(y,0,v,\vec{z})H_4(\vec{\zeta}_n+\vec{\xi}_n,v,\vec{z})H_2\left(\frac{|\lambda|^2}{2\mathrm{Re}\lambda},v,\vec{z}\right)\\
&\qquad\times H_5\left(\frac{|\lambda|^2}{2\mathrm{Re}\lambda},v,\vec{z}\right)d\sigma(v)d\rho(\vec{z})
\end{aligned}
$$

so that we have

$$
\begin{aligned}
&\|T_{\overline{\lambda}}[T_\lambda[\Psi|X_n](\cdot,\vec{\xi}_n)|X_n](\cdot,\vec{\zeta}_n)-\Psi_1(\cdot,\vec{\zeta}_n+\vec{\xi}_n)\|_p\\
&\le\int_{\mathbb{R}^r}\int_{L_2[0,t]}\left[1-H_2\left(\frac{|\lambda|^2}{2\mathrm{Re}\lambda},v,\vec{z}\right)H_5\left(\frac{|\lambda|^2}{2\mathrm{Re}\lambda},v,\vec{z}\right)\right]d|\sigma|(v)d|\rho|(\vec{z})
\end{aligned}
$$

which converges to 0 as λ approaches to $-iq$ through $\mathbb{C}_+$ by the dominated convergence theorem. □

THEOREM 4.3. *Let X_{n+1} be given by (1.3) and let $1\le p\le\infty$. Furthermore, let Ψ_1 and Ψ_2 be as given in Theorem 2.3. Then for a nonzero real q, w_φ-a.e. $y\in C[0,t]$ and $P_{X_{n+1}}$-a.e. $\vec{\xi}_{n+1},\vec{\zeta}_{n+1}\in\mathbb{R}^{n+2}$,*

$$
\begin{aligned}
&T_q^{(p)}[[(\Psi_1*\Psi_2)_q|X_{n+1}](\cdot,\vec{\xi}_{n+1})|X_{n+1}](y,\vec{\zeta}_{n+1})\\
&=\left[T_q^{(p)}[\Psi_1|X_{n+1}]\left(\frac{1}{\sqrt{2}}y,\frac{1}{\sqrt{2}}(\vec{\zeta}_{n+1}+\vec{\xi}_{n+1})\right)\right]\\
&\qquad\times\left[T_q^{(p)}[\Psi_2|X_{n+1}]\left(\frac{1}{\sqrt{2}}y,\frac{1}{\sqrt{2}}(\vec{\zeta}_{n+1}-\vec{\xi}_{n+1})\right)\right].
\end{aligned}\tag{4.1}
$$

Proof. Let $\lambda \in \mathbb{C}_+^{\sim}$. For $\lambda_1 > 0$, w_φ-a.e. $y \in C[0,t]$ and $P_{X_{n+1}}$-a.e. $\vec{\xi}_{n+1}, \vec{\zeta}_{n+1} \in \mathbb{R}^{n+2}$,

$$\begin{aligned}
&I^{\lambda_1}_{[(\Psi_1 * \Psi_2)_\lambda | X_{n+1}](\cdot, \vec{\xi}_{n+1})}(y, \vec{\zeta}_{n+1}) \\
&= \int_{\mathbb{R}^r}\int_{\mathbb{R}^r}\int_{L_2[0,t]}\int_{L_2[0,t]}\int_C H_1\bigg(\lambda_1^{-\frac{1}{2}}(x-[x]) + y + [\vec{\zeta}_{n+1}], \vec{\xi}_{n+1}, \\
&\frac{1}{\sqrt{2}}v_1, \frac{1}{\sqrt{2}}\vec{z}_1\bigg) H_1\bigg(\lambda_1^{-\frac{1}{2}}(x-[x]) + y + [\vec{\zeta}_{n+1}], -\vec{\xi}_{n+1}, \frac{1}{\sqrt{2}}v_2, \frac{1}{\sqrt{2}}\vec{z}_2\bigg) \\
&\times H_2\bigg(\lambda, \frac{1}{\sqrt{2}}(v_1 - v_2), \frac{1}{\sqrt{2}}(\vec{z}_1 - \vec{z}_2)\bigg) dw_\varphi(x) d\sigma_1(v_1) d\sigma_2(v_2) d\rho_1(\vec{z}_1) d\rho_2(\vec{z}_2)
\end{aligned}$$

by Theorem 2.3, where H_1 and H_2 are given by (2.3) and (2.4), respectively. Now, we have

$$\begin{aligned}
&I^{\lambda_1}_{[(\Psi_1 * \Psi_2)_\lambda | X_{n+1}](\cdot, \vec{\xi}_{n+1})}(y, \vec{\zeta}_{n+1}) \\
&= \int_{\mathbb{R}^r}\int_{\mathbb{R}^r}\int_{L_2[0,t]}\int_{L_2[0,t]}\int_C H_1\bigg(\frac{1}{\sqrt{2}}y, \frac{1}{\sqrt{2}}(\vec{\zeta}_{n+1} + \vec{\xi}_{n+1}), v_1, \vec{z}_1\bigg) \\
&\times H_1\bigg(\frac{1}{\sqrt{2}}y, \frac{1}{\sqrt{2}}(\vec{\zeta}_{n+1} - \vec{\xi}_{n+1}), v_2, \vec{z}_2\bigg) H_2\bigg(\lambda, \frac{1}{\sqrt{2}}(v_1 - v_2), \frac{1}{\sqrt{2}}(\vec{z}_1 - \vec{z}_2)\bigg) \\
&\times H_3\bigg(\lambda_1, \frac{1}{\sqrt{2}}(v_1 + v_2), \frac{1}{\sqrt{2}}(\vec{z}_1 + \vec{z}_2), x\bigg) \\
&\quad dw_\varphi(x) d\sigma_1(v_1) d\sigma_2(v_2) d\rho_1(\vec{z}_1) d\rho_2(\vec{z}_2),
\end{aligned}$$

where H_3 is given by (2.6). By Lemma 2.1, we obtain that

$$\begin{aligned}
&I^{\lambda_1}_{[(\Psi_1 * \Psi_2)_\lambda | X_{n+1}](\cdot, \vec{\xi}_{n+1})}(y, \vec{\zeta}_{n+1}) \\
&= \int_{\mathbb{R}^r}\int_{\mathbb{R}^r}\int_{L_2[0,t]}\int_{L_2[0,t]} H_1\bigg(\frac{1}{\sqrt{2}}y, \frac{1}{\sqrt{2}}(\vec{\zeta}_{n+1} + \vec{\xi}_{n+1}), v_1, \vec{z}_1\bigg) \\
&\times H_1\bigg(\frac{1}{\sqrt{2}}y, \frac{1}{\sqrt{2}}(\vec{\zeta}_{n+1} - \vec{\xi}_{n+1}), v_2, \vec{z}_2\bigg) H_2\bigg(\lambda, \frac{1}{\sqrt{2}}(v_1 - v_2), \\
&\frac{1}{\sqrt{2}}(\vec{z}_1 - \vec{z}_2)\bigg) H_2\bigg(\lambda_1, \frac{1}{\sqrt{2}}(v_1 + v_2), \frac{1}{\sqrt{2}}(\vec{z}_1 + \vec{z}_2)\bigg) d\sigma_1(v_1) \\
&d\sigma_2(v_2) d\rho_1(\vec{z}_1) d\rho_2(\vec{z}_2).
\end{aligned} \tag{4.2}$$

By (2.5), the Morera's theorem and the dominated convergence theorem, we have the analytic extension $T_{\lambda_1}[[(\Psi_1 * \Psi_2)_\lambda | X_{n+1}](\cdot, \vec{\xi}_{n+1}) | X_{n+1}](y, \vec{\zeta}_{n+1})$ of (4.2) as function of $\lambda_1 \in \mathbb{C}_+$. Let $T_q^{(p)}[[(\Psi_1 * \Psi_2)_\lambda | X_{n+1}](\cdot, \vec{\xi}_{n+1}) | X_{n+1}](y, \vec{\zeta}_{n+1})$ be given by the right hand side of (4.2), where λ_1

is replaced by $-iq$, and let $\frac{1}{p}+\frac{1}{p'}=1$. By (2.5)

$$\begin{aligned}
&\|T_{\lambda_1}[[(\Psi_1*\Psi_2)_\lambda|X_{n+1}](\cdot,\vec{\xi}_{n+1})|X_{n+1}](\cdot,\vec{\zeta}_{n+1})\\
&\qquad -T_q^{(p)}[[(\Psi_1*\Psi_2)_\lambda|X_{n+1}](\cdot,\vec{\xi}_{n+1})|X_{n+1}](\cdot,\vec{\zeta}_{n+1})\|_{p'}\\
&\le \int_{\mathbb{R}^r}\int_{\mathbb{R}^r}\int_{L_2[0,t]}\int_{L_2[0,t]}\bigg|H_2\bigg(\lambda_1,\frac{1}{\sqrt{2}}(v_1+v_2),\frac{1}{\sqrt{2}}(\vec{z}_1+\vec{z}_2)\bigg)-\\
&H_2\bigg(-iq,\frac{1}{\sqrt{2}}(v_1+v_2),\frac{1}{\sqrt{2}}(\vec{z}_1+\vec{z}_2)\bigg)\bigg|d|\sigma_1|(v_1)d|\sigma_2|(v_2)d|\rho_1|(\vec{z}_1)d|\rho_2|(\vec{z}_2)
\end{aligned}$$

which converges to 0 as λ_1 approaches to $-iq$ through $\mathbb{C}_+$ by the dominated convergence theorem. This shows the existence of $T_q^{(p)}[[(\Psi_1*\Psi_2)_\lambda|X_{n+1}](\cdot,\vec{\xi}_{n+1})|X_{n+1}](y,\vec{\zeta}_{n+1})$. Now, we have by (4.2)

$$\begin{aligned}
&T_q^{(p)}[[(\Psi_1*\Psi_2)_q|X_{n+1}](\cdot,\vec{\xi}_{n+1})|X_{n+1}](y,\vec{\zeta}_{n+1})\\
&=\int_{\mathbb{R}^r}\int_{\mathbb{R}^r}\int_{L_2[0,t]}\int_{L_2[0,t]}H_1\bigg(\frac{1}{\sqrt{2}}y,\frac{1}{\sqrt{2}}(\vec{\zeta}_{n+1}+\vec{\xi}_{n+1}),v_1,\vec{z}_1\bigg)H_1\bigg(\\
&\frac{1}{\sqrt{2}}y,\frac{1}{\sqrt{2}}(\vec{\zeta}_{n+1}-\vec{\xi}_{n+1}),v_2,\vec{z}_2\bigg)H_2(-iq,v_1,\vec{z}_1))H_2(-iq,v_2,\vec{z}_2)\\
&d\sigma_1(v_1)d\sigma_2(v_2)d\rho_1(\vec{z}_1)d\rho_2(\vec{z}_2)
\end{aligned}$$

which completes the proof by Theorem 2.2. □

Note that by the same method as used in the proof of Theorem 4.3, we can obtain (4.1), where $-iq$ is replaced by $\lambda\in\mathbb{C}_+$.

Now, we have the final result of our work.

THEOREM 4.4. *If X_{n+1}, $\vec{\xi}_{n+1}$ and $\vec{\zeta}_{n+1}$ in Theorem 4.3 are replaced by X_n, $\vec{\xi}_n\in\mathbb{R}^{n+1}$ and $\vec{\zeta}_n\in\mathbb{R}^{n+1}$, respectively, then the conclusion given by (4.1) is still true.*

Proof. Let $\lambda\in\tilde{\mathbb{C}}_+$. For $\lambda_1>0$, w_φ-a.e. $y\in C[0,t]$ and P_{X_n}-a.e. $\vec{\xi}_n,\vec{\zeta}_n\in\mathbb{R}^{n+1}$,

$$\begin{aligned}
&K^{\lambda_1}_{[(\Psi_1*\Psi_2)_\lambda|X_n](\cdot,\vec{\xi}_n)}(y,\vec{\zeta}_n)\\
&=\int_{\mathbb{R}^r}\int_{\mathbb{R}^r}\int_{L_2[0,t]}\int_{L_2[0,t]}H_1\bigg(y,0,\frac{1}{\sqrt{2}}(v_1+v_2),\frac{1}{\sqrt{2}}(\vec{z}_1+\vec{z}_2)\bigg)H_2\bigg(\lambda_1,
\end{aligned}$$

$$\frac{1}{\sqrt{2}}(v_1+v_2),\frac{1}{\sqrt{2}}(\vec{z}_1+\vec{z}_2)\Big)H_4\Big(\vec{\zeta}_n,\frac{1}{\sqrt{2}}(v_1+v_2),\frac{1}{\sqrt{2}}(\vec{z}_1+\vec{z}_2)\Big)H_5$$
$$\Big(\lambda_1,\frac{1}{\sqrt{2}}(v_1+v_2),\frac{1}{\sqrt{2}}(\vec{z}_1+\vec{z}_2)\Big)H_2\Big(\lambda,\frac{1}{\sqrt{2}}(v_1-v_2),\frac{1}{\sqrt{2}}(\vec{z}_1-\vec{z}_2)\Big)$$
$$H_4\Big(\vec{\xi}_n,\frac{1}{\sqrt{2}}(v_1-v_2),\frac{1}{\sqrt{2}}(\vec{z}_1-\vec{z}_2)\Big)H_5\Big(\lambda,\frac{1}{\sqrt{2}}(v_1-v_2),\frac{1}{\sqrt{2}}(\vec{z}_1-\vec{z}_2)\Big)$$
$$d\sigma_1(v_1)d\sigma_2(v_2)d\rho_1(\vec{z}_1)d\rho_2(\vec{z}_2)$$

by Lemma 3.1, Theorems 3.2 and 3.3, where H_1, H_2, H_4 and H_5 are given by (2.3), (2.4), (3.1) and (3.2), respectively. By (2.5), the Morera's theorem and the dominated convergence theorem, we have the analytic extension $T_{\lambda_1}[[(\Psi_1*\Psi_2)_\lambda|X_n](\cdot,\,\vec{\xi}_n)|X_n](y,\,\vec{\zeta}_n)$ of $K^{\lambda_1}_{[(\Psi_1*\Psi_2)_\lambda|X_n](\cdot,\vec{\xi}_n)}(y,\vec{\zeta}_n)$ as function of $\lambda_1\in\mathbb{C}_+$. Let $T_q^{(p)}[[(\Psi_1*\Psi_2)_\lambda|X_n](\cdot,\vec{\xi}_n)|X_n](y,\vec{\zeta}_n)$ be given by the right-hand side of the above equality, where λ_1 is replaced by $-iq$, and let $\frac{1}{p}+\frac{1}{p'}=1$. Then we have

$$\begin{aligned}&\|T_{\lambda_1}[[(\Psi_1*\Psi_2)_\lambda|X_n](\cdot,\vec{\xi}_n)|X_n](\cdot,\vec{\zeta}_n)\\&\qquad -T_q^{(p)}[[(\Psi_1*\Psi_2)_\lambda|X_n](\cdot,\vec{\xi}_n)|X_n](\cdot,\vec{\zeta}_n)\|_{p'}\\&\le\int_{\mathbb{R}^r}\int_{\mathbb{R}^r}\int_{L_2[0,t]}\int_{L_2[0,t]}\Big|H_2\Big(\lambda_1,\frac{1}{\sqrt{2}}(v_1+v_2),\frac{1}{\sqrt{2}}(\vec{z}_1+\vec{z}_2)\Big)\\&\times H_5\Big(\lambda_1,\frac{1}{\sqrt{2}}(v_1+v_2),\frac{1}{\sqrt{2}}(\vec{z}_1+\vec{z}_2)\Big)-H_2\Big(-iq,\frac{1}{\sqrt{2}}(v_1+v_2),\\&\frac{1}{\sqrt{2}}(\vec{z}_1+\vec{z}_2)\Big)H_5\Big(-iq,\frac{1}{\sqrt{2}}(v_1+v_2),\frac{1}{\sqrt{2}}(\vec{z}_1+\vec{z}_2)\Big)\Big|\\&d|\sigma_1|(v_1)d|\sigma_1|(v_2)d|\rho_1|(\vec{z}_1)d|\rho_2|(\vec{z}_2)\end{aligned}$$

which converges to 0 as λ approaches to $-iq$ through $\mathbb{C}_+$ by the dominated convergence theorem. This shows the existence of $T_q^{(p)}[[(\Psi_1*\Psi_2)_\lambda|X_n](\cdot,\vec{\xi}_n)|\,X_n](\cdot,\vec{\zeta}_n)$. By (2.3) and (3.1), it is not difficult to show

$$\begin{aligned}&H_1\Big(y,0,\frac{1}{\sqrt{2}}(v_1+v_2),\frac{1}{\sqrt{2}}(\vec{z}_1+\vec{z}_2)\Big)\\&=H_1\Big(\frac{1}{\sqrt{2}}y,0,v_1,\vec{z}_1\Big)H_1\Big(\frac{1}{\sqrt{2}}y,0,v_2,\vec{z}_2\Big)\end{aligned}$$

and

$$\begin{aligned}
&H_4\bigg(\vec{\zeta}_n, \frac{1}{\sqrt{2}}(v_1+v_2), \frac{1}{\sqrt{2}}(\vec{z}_1+\vec{z}_2)\bigg)\\
&\times H_4\bigg(\vec{\xi}_n, \frac{1}{\sqrt{2}}(v_1-v_2), \frac{1}{\sqrt{2}}(\vec{z}_1-\vec{z}_2)\bigg)\\
&= H_4\bigg(\frac{1}{\sqrt{2}}(\vec{\zeta}_n+\vec{\xi}_n), v_1, \vec{z}_1\bigg)H_4\bigg(\frac{1}{\sqrt{2}}(\vec{\zeta}_n-\vec{\xi}_n), v_2, \vec{z}_2\bigg).
\end{aligned}$$

Furthermore, by (2.4) and (3.2),

$$\begin{aligned}
&H_2\bigg(\lambda, \frac{1}{\sqrt{2}}(v_1+v_2), \frac{1}{\sqrt{2}}(\vec{z}_1+\vec{z}_2)\bigg)\\
&\times H_2\bigg(\lambda, \frac{1}{\sqrt{2}}(v_1-v_2), \frac{1}{\sqrt{2}}(\vec{z}_1-\vec{z}_2)\bigg) = H_2(\lambda, v_1, \vec{z}_1)H_2(\lambda, v_2, \vec{z}_2)
\end{aligned}$$

and

$$\begin{aligned}
&H_5\bigg(\lambda, \frac{1}{\sqrt{2}}(v_1+v_2), \frac{1}{\sqrt{2}}(\vec{z}_1+\vec{z}_2)\bigg)\\
&\times H_5\bigg(\lambda, \frac{1}{\sqrt{2}}(v_1-v_2), \frac{1}{\sqrt{2}}(\vec{z}_1-\vec{z}_2)\bigg) = H_5(\lambda, v_1, \vec{z}_1)H_5(\lambda, v_2, \vec{z}_2).
\end{aligned}$$

Now, we have the result by Theorem 3.2. □

Note that by the same method as used in the proof of Theorem 4.4, we can show the same equality in the above theorem, where $-iq$ is replaced by $\lambda \in \mathbb{C}_+$.

References

[1] R. H. Cameron and D. A. Storvick, *Some Banach algebras of analytic Feynman integrable functionals,* Lecture Notes in Mathematics **798**, Springer, Berlin-New York, 1980.

[2] K. S. Chang, D. H. Cho, B. S. Kim, T. S. Song and I. Yoo, *Conditional Fourier-Feynman transform and convolution product over Wiener paths in abstract Wiener space,* Integral Transforms Spec. Funct. **14** (2003), no. 3, 217-235.

[3] S. J. Chang and D. Skoug, *The effect of drift on conditional Fourier-Feynman transforms and conditional convolution products,* Int. J. Appl. Math. **2** (2000), no. 4, 505-527.

[4] D. H. Cho, *A time-independent conditional Fourier-Feynman transform and convolution product on an analogue of Wiener space,* Honam J. Math. (2013), to appear.

[5] D. H. Cho, *A time-dependent conditional Fourier-Feynman transform and convolution product on an analogue of Wiener space,* Houston J. Math. (2012), submitted.

[6] D. H. Cho, *A conditional integral transform and conditional convolution product on a function space,* Integral Transforms Spec. Funct. **23** (2012), no 6, 405-420.

[7] D. H. Cho, *A simple formula for an analogue of conditional Wiener integrals and its applications II,* Czechoslovak Math. J. **59** (2009), no. 2, 431-452.

[8] D. H. Cho, *A simple formula for an analogue of conditional Wiener integrals and its applications,* Trans. Amer. Math. Soc. **360** (2008), no. 7, 3795-3811.

[9] D. H. Cho, *Conditional Fourier-Feynman transform and convolution product over Wiener paths in abstract Wiener space: an L_p theory,* J. Korean Math. Soc. **41** (2004), no. 2, 265-294.

[10] D. H. Cho, B. J. Kim and I. Yoo, *Analogues of conditional Wiener integrals and their change of scale transformations on a function space,* J. Math. Anal. Appl. **359** (2009), no. 2, 421-438.

[11] M. K. Im and K. S. Ryu, *An analogue of Wiener measure and its applications,* J. Korean Math. Soc. **39** (2002), no. 5, 801-819.

[12] M. J. Kim, *Conditional Fourier-Feynman transform and convolution product on a function space,* Int. J. Math. Anal. **3** (2009), no. 10, 457-471.

[13] K. S. Ryu and M. K. Im, *A measure-valued analogue of Wiener measure and the measure-valued Feynman-Kac formula,* Trans. Amer. Math. Soc. **354** (2002), no. 12, 4921-4951.

[14] K. S. Ryu, M. K. Im, and K. S. Choi, *Survey of the theories for analogue of Wiener measure space,* Interdiscip. Inform. Sci. **15** (2009), no. 3, 319-337.

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